5. V. N. Pribylov and L. T. Chernyi, "Electrification of dielectric liquids flowing in tubes," Izv. Akad. Nauk SSSR, Mekh. Zhid. Gaza No. 6 (1979).
6. L. B. Leb, Static Electrification [in Russian], Gostekhizdat, Moscow–Leningrad (1963).
7. Ya. I. Gerasimov et al., Course in Physical Chemistry [in Russian], Vol. 2, Khimiya, Moscow (1973).
8. I. A. Slezkin, Dynamics of a Viscous Incompressible Fluid [in Russian], Gostekhizdat, Moscow (1955).
9. D. S. Kuznetsov, Special Functions [in Russian], Vysshaya Shkola Moscow (1962).
10. Yu. K. Kopylov, "Fundamental laws of the electrical conductivity of high-resistance organic liquids for an activation character of contact processes," Organic Semiconducting Liquids, Vol. 27 [in Russian], Trudy Dnepropetrovosk Inst. (1974).
11. A. I. Zhakin, "Some problems of the electrohydrodynamic stability and the electroconvection of incompressible fluids," Author's Abstract of Candidate's Dissertation, Kharkov State Univ. (1980).
12. N. L. Glinka, General Chemistry [Russian translation], Khimiya, Leningrad (1972).
13. L. S. Kazatskaya and I. M. Solodovnichenko, "On the role of electroinduction effects of molecules in the production of charge carriers in an organic liquid," Elektron. Obrab. Mater., No. 2 (1979).

INITIAL ASYMPTOTE TO THE SOLUTION OF THE PROBLEM OF DROPLET INCIDENCE ON A PLANE

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The initial stage of collision of a spherical droplet on a solid plane is considered. It is assumed that the droplet liquid is ideal and incompressible, and that surface tension and external mass forces are absent.

This problem is closely related to that of entry of a blunt body into a liquid, which was first considered in [1]. The method for calculation of the resistive forces, developed in [1], is based on the assumption that the velocity distribution on the free surface at each moment is the same as that obtained directly after collision of a floating plate of the same dimensions.

These problems have the following unique features: 1) the flow region Ω_t is unknown; 2) the contact line between free liquid surface and the solid must be determined at the boundary of the flow region; 3) singularities may appear in the solution on this line.

A new approach to problems of this kind is the introduction of Lagrangian coordinates [2, 3], in which the flow region is fixed.

1. At time t = 0 a liquid sphere of radius a is tangent upon a solid plane, which moves along the z axis at velocity v. We must find the liquid motion which then occurs. In the space formed by Lagrangian Cartesian coordinates ξ, η, ζ the region occupied by the liquid is known, being a sphere of radius a with center at the origin. We denote this region by Ω_0. The variables x, y, z denote the corresponding Euler coordinates, Γ is the free surface of the liquid, and Σ is the contact spot between droplet and solid plane. The Euler equations, written in Lagrangian coordinates, have the form [3]

$$M_0^*\mathbf{x}_{tt}+\frac{1}{\gamma}\nabla_{\xi}p=0,\quad \det M_0=1 \text{ in } \Omega_0 \tag{1.1}$$

with boundary conditions $p|_\Gamma=0$, $z_t|_\Sigma=v$ and initial conditions $\mathbf{x}|_{t=0}=\xi$, $\mathbf{x}_t|_{t=0}$, where $\mathbf{x}=(x, y, z)$; $\xi=(\xi, \eta, \zeta)$; $M_0=\partial(\mathbf{x})/\partial(\xi)$; M_0^* is the matrix conjugate to M_0 and p is the pressure. The problem is a complex one because of its nonlinearity and the existence of the unknown line on the sphere boundary $\partial\Omega_0$, dividing Γ and Σ.

2. We will linearize Eq. (1.1) for the initial rest state, keeping terms of zeroth- and first-order smallness in displacement. For the linearized problem we can introduce a displacement potential $\Phi=\Phi(\xi, \eta, \zeta, t)$, which in view of the continuity equation, will be a function harmonic in Ω_0. From the momentum equation follows that

$$p=-\gamma\Phi_{tt} \tag{2.1}$$

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thus relating the pressure and the displacement potential. With consideration of this expression, after double integration over t and use of the initial conditions, the condition on Γ may be written as $\Phi|_{\Gamma}=0$. The nonflow condition takes on the form $\Phi_{zt}|_{\Sigma}=-v$.

We introduce Lagrangian spherical coordinates ρ, θ, φ such that $\xi=\rho\sin\theta\cos\varphi$, $\eta=\rho\sin\theta\sin\varphi$, $\zeta=\rho\cos\theta$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\varphi\leqslant 2\pi$. The problem formulation is invariant relative to rotation about the z axis, permitting search for a function Φ which is independent of φ. Then in Lagrangian coordinates the contact line between the liquid free surface and the solid wall will be specified by the expression $\theta=\theta_0(t)$, $\rho=a$, where $\theta_0(t)$ must be determined in the course of solving the boundary problem for Φ. We write the nonflow condition in spherical coordinates and integrate it over t:

$$(\partial\Phi/\partial\rho)\cos\theta+(1/a)(\partial\Phi/\partial\theta)\sin\theta=-vt+a(1-\cos\theta),\ \rho=a,$$
$$0\leqslant\theta<\theta_0(t).$$

We divide the left and right sides of the expression by cos θ and express the right side in terms of first and second order Legendre polynomials:

$$\frac{\partial\Phi}{\partial\rho}+\frac{1}{a}\frac{\partial\Phi}{\partial\theta}\operatorname{tg}\theta=\left(\frac{a}{2}-2vt\right)P_1(\cos\theta)+\left(vt-\frac{a}{2}\right)P_2(\cos\theta)+O\left(t\theta_0^2\right),$$
$$\rho=a,\quad 0\leqslant\theta<\theta_0(t).$$

The term $a^{-1}\Phi_\theta\tan\theta$ is quadratically small in comparison to Φ_ρ, since as $t\to 0$ $\theta_0(t)$, and consequently θ also vanishes, so that within the framework of linear theory it may be dropped together with $O(t\theta_0^2)$. As a result, we obtain a mixed boundary problem for a function harmonic at $\rho<a$:

$$\Delta\Phi=0,\ \rho<a,\ \Phi=0,\ \rho=a,\ \theta_0(t)<\theta\leqslant\pi,$$
$$\Phi_\rho=(a/2-2vt)P_1(\cos\theta)+(vt-a/2)P_2(\cos\theta),\ \rho=a,\ 0\leqslant\theta<\theta_0(t). \tag{2.2}$$

Moreover, we must specify that particles of the free surface cannot penetrate beyond the solid wall. This limitation can be written as an inequality

$$\Phi_\rho\leqslant(a-vt)/\cos\theta-a,\ \rho=a,\ \theta_0(t)<\theta\leqslant\pi/2, \tag{2.3}$$

which must be verified after solution of the problem.

We introduce the dimensionless variables r, τ and a new function $u(r,\theta,\tau)=(2/a^2)\Phi(ar,\ \theta,\ a\tau/2v)-(1-2\tau)\ r\ \ P_1(\cos\theta)-(1/2)(\tau-1)r^2P_2(\cos\theta)$, $r=\rho/a$, $\tau=2vt/a$. From Eq. (2.2) it follows that

$$\Delta u=0,\ r<1,\ u=(2\tau-1)P_1(\cos\theta)+(1/2)(1-\tau)P_2(\cos\theta)\equiv f(\theta,\tau),$$
$$r=1,\ \theta_0<\theta\leqslant\pi,\ u_r=0,\ r=1,\ 0\leqslant\theta<\theta_0. \tag{2.4}$$

We will seek a solution of this problem in the form of a series in Legendre polynomials

$$u(r,\theta,\tau)=\sum_{n=0}^{\infty}A_n(\tau)r^nP_n(\cos\theta). \tag{2.5}$$

Using the boundary conditions, we arrive at:

$$\sum_{n=0}^{\infty}A_n(\tau)P_n(\cos\theta)=f(\theta,\tau),\quad \theta_0<\theta\leqslant\pi,$$
$$\sum_{n=0}^{\infty}nA_n(\tau)P_n(\cos\theta)=0,\quad 0\leqslant\theta<\theta_0, \tag{2.6}$$

which are termed paired series [4]. It will be conveneint to introduce new coefficient $C_n=2nA_n/(2n+1)$, $n\geq 1$ and $C_0=0$, and the numbers $g_n=-1/2n$, $n\geq 1$, and g_0, any number. The paired series written in terms of C_n, g_n take on a form which is termed standard:

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)C_nP_n(\cos\theta)=0,\quad 0\leqslant\theta<\theta_0; \tag{2.7a}$$

$$\sum_{n=0}^{\infty}(1-g_n)C_nP_n(\cos\theta)=f(\theta,\tau)-A_0(\tau),\quad \theta_0<\theta\leqslant\pi. \tag{2.7b}$$

If C_n (and thus A_n) is determined by solution of Eq. (2.7), then the formal solution of Eq. (2.4) will be given by the series of Eq. (2.5). System (2.7) can be reduced to a regular Fredholm integral equation by a method based on the Meler–Dirichlet integral representation of Legendre polynomials

$$P_n(\cos\theta)=\frac{2}{\pi}\int_{\theta}^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sqrt{2(\cos\theta-\cos x)}}dx$$

and expansions of certain discontinuous functions in series in Legendre polynomials [4]:

$$\sum_{n=0}^{\infty} \sin\left(n+\frac{1}{2}\right) tP_n(\cos\theta) = \begin{cases} 0, & 0 \leqslant t < \theta < \pi, \\ [2(\cos\theta - \cos t)]^{-1/2}, & 0 < \theta < t \leqslant \pi; \end{cases} \tag{2.8}$$

$$\sum_{n=0}^{\infty} \cos\left(n+\frac{1}{2}\right) tP_n(\cos\theta) = \begin{cases} [2(\cos t - \cos\theta)]^{-1/2}, & 0 \leqslant t < \theta < \pi, \\ 0, & 0 < \theta < t \leqslant \pi. \end{cases} \tag{2.9}$$

We take

$$C_n(\tau) = \int_{\theta_0(\tau)}^{\pi} \varphi(y, \tau) \sin\left(n+\frac{1}{2}\right) y dy, \tag{2.10}$$

where $\varphi(x, \tau)$ is a new unknown function, which is assumed to be continuously differentiable with respect to x and continually differentiable twice with respect to τ. If we now substitute Eq. (2.10) into Eq. (2.7a), by using Eq. (2.9) we obtain an identity [4]. Then, substituting Eq. (2.10) in Eq. (2.7b) we obtain a Fredholm equation for $\varphi(x, \tau)$

$$\varphi(x, \tau) - \frac{1}{\pi} \int_{\theta_0}^{\pi} K(x, y) \varphi(y, \tau) dy = \frac{2}{\pi} \frac{d}{dx} \int_{x}^{\pi} \frac{(f(\theta, \tau) - A_0(\tau)) \sin\theta d\theta}{\sqrt{2(\cos x - \cos\theta)}}, \quad \theta_0 < x \leqslant \pi, \tag{2.11}$$

the integrand of which has the form

$$K(x, y) = 2 \sum_{n=0}^{\infty} g_n \sin\left(n+\frac{1}{2}\right) x \sin\left(n+\frac{1}{2}\right) y.$$

We calculate the right side of Eq. (2.11) and write the Fredholm equation in final form, using the condition $C_0(\tau) = 0$:

$$\varphi(x, \tau) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(n+\frac{1}{2}\right) x \int_{\theta_0}^{\pi} \varphi(y, \tau) \sin\left(n+\frac{1}{2}\right) y dy =$$

$$= \frac{1}{\pi}\left(\frac{1-\tau}{2} + 2A_0(\tau)\right) \sin\frac{x}{2} + \frac{2}{\pi}(1-2\tau)\sin\frac{3x}{2} + \frac{1}{\pi}(\tau - 1)\sin\frac{5x}{2} \equiv \psi(x, \tau).$$

The coefficient $A_0(\tau)$ is determined after solution of this equation from the condition $C_0 = 0$.

3. We will determine the motion of the free surface. Since $\Phi|_\Gamma = 0$, only the normal component of the displacement vector $\Phi_\rho|_\Gamma$ will be nonzero:

$$\Phi_\rho = (a/2)u_r + (a/2 - 2vt)P_1(\cos\theta) + (vt - a/2)P_2(\cos\theta).$$

It follows from Eq. (2.5) that

$$u_r|_\Gamma = \sum_{n=0}^{\infty} nA_n(\tau) P_n(\cos\theta) = \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right) C_n(\tau) P_n(\cos\theta).$$

Within this expression we substitute the expression for $C_n(\tau)$ in terms of $\varphi(x, \tau)$ and use the discontinuous series (2.8), (2.9):

$$u_r|_\Gamma = \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right) \int_{\theta_0}^{\pi} \varphi(y, \tau) \sin\left(n+\frac{1}{2}\right) yP_n(\cos\theta) dy = -\int_{\theta_0}^{\pi} \varphi(y, \tau) \sum_{n=0}^{\infty} P_n(\cos\theta)\, d\cos\left(n+\frac{1}{2}\right) y =$$

$$+ \int_{\theta_0}^{\pi} \left(\sum_{n=0}^{\infty} \cos\left(n+\frac{1}{2}\right) yP_n(\cos\theta)\right) \varphi_y(y, \tau) dy = \varphi(\theta_0(\tau), \tau) \sum_{n=0}^{\infty} \cos\left(n+\frac{1}{2}\right) \theta_0 P_n(\cos\theta)$$

$$= \frac{\varphi(\theta_0(\tau), \tau)}{\sqrt{2(\cos\theta_0 - \cos\theta)}} + \int_{\theta_0}^{\theta} \frac{\varphi_y(y, \tau) dy}{\sqrt{2(\cos y - \cos\theta)}}.$$

From condition (2.3) we obtain an equation for determination of $\theta_0(\tau)$:

$$\varphi(\theta_0(\tau), \tau) = 0. \tag{3.1}$$

The pressure is determined with Eq. (2.1). Using the relationships $\Phi(\rho, \theta, t)$ and $u(r, \theta, \tau)$, we write

$$p = -2\gamma u_{\tau\tau}(1, \theta, \tau), \; 0 \leqslant \theta < \theta_0(\tau). \tag{3.2}$$

Applying Eqs. (2.8), (2.9), after various transformations we obtain

$$u_{\tau\tau}(1, \theta, \tau)|_\Sigma = -\left.\frac{\varphi_\tau(x, \tau)\,\theta_0'(\tau)}{\sqrt{2(\cos\theta - \cos\theta_0)}}\right|_{x=\theta_0} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[P_n(\cos\theta) - P_n(\cos\theta_0)]\int_{\theta_0}^{\pi}\varphi_{\tau\tau}(x, \tau)\sin\left(n + \frac{1}{2}\right)x\,dx, \tag{3.3}$$

the series in which converges absolutely, since $P_n(\cos\theta) = O(n^{-1/2})$ as $n \to \infty$. The force of the droplet collision with the plate is determined by integrating the pressure over the contact spot area

$$F = 2\pi a^2 \int_0^{\theta_0} p(1, \theta, \tau)\sin\theta\,d\theta$$

or, using Eqs. (3.2), (3.3), we have

$$F = 8\pi\gamma a^2 v^2 \varphi_\tau(x, \tau)\,\theta_0'(\tau)|_{x=\theta_0(\tau)}\sin\frac{\theta_0}{2} + \tag{3.4}$$

$$+ 4\gamma a^2 v^2 \sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^{\theta_0}[P_n(\cos\theta_0) - P_n(\cos\theta)]\sin\theta\,d\theta\right)\left(\int_{\theta_0}^{\pi}\varphi_{\tau\tau}(x,\tau)\sin\left(n + \frac{1}{2}\right)x\,dx\right).$$

4. We will find an approximate solution of integral equation (2.11). We expand the unknown function $\varphi(x, \tau)$, the dimensionless time τ, the coefficient $A_0(\tau)$ and $\psi(x, \tau)$ in series in θ_0:

$$\varphi(x, \tau) = \varphi_0(x) + \varphi_1(x)\theta_0 + \ldots, \quad \psi(x, \tau) = \psi_0(x) + \psi_1(x)\theta_0 + \ldots,$$
$$A_0(\tau) = b_0 + b_1\theta_0 + b_2\theta_0^2 + \ldots, \quad \tau = a_2\theta_0^2 + a_3\theta_0^3 + \ldots \tag{4.1}$$

We assume that the function $\varphi(x, \tau)$ is analytic over the interval $[0, \pi]$. Then Eq. (2.11) can be written in the following manner:

$$\varphi(x, \tau) - \frac{1}{\pi}\int_0^{\pi}K(x, y)\varphi(y, \tau)\,dy + \frac{1}{\pi}\int_0^{\theta_0}K(x, y)\varphi(y, \tau)\,dy = \psi(x, \tau), \quad \theta_0 < x \leqslant \pi. \tag{4.2}$$

We expand the integrand $K(x, y)$ using Taylor's formula for $y \in [0, \theta_0)$:

$$K(x, y) = -\frac{\pi}{2}\sin\frac{x}{2} + \left(\frac{1}{2}\sin\frac{x}{2}\ln\left(2\sin\frac{x}{2}\right) - \frac{1}{2}\operatorname{ctg}\frac{x}{2}\cos\frac{x}{2} + \frac{1}{2}\sin\frac{x}{2} + \frac{1}{4}x\cos x\right)y + O(y^2).$$

The equation for determination of $\varphi_0(x)$ follows from Eq. (2.11) at $\tau = 0$.

$$\varphi_0(x) - \frac{1}{\pi}\int_0^{\pi}K(x, y)\varphi_0(y)\,dy = \psi_0(x).$$

We seek a solution of this equation in the form

$$\varphi_0(x) = \sum_{n=0}^{\infty}\alpha_n\sin\left(n + \frac{1}{2}\right)x.$$

It can be shown that

$$\varphi_0(x) = \left(-\frac{1}{2\pi} - \frac{2}{\pi}b_0\right)\sin\frac{x}{2} - \frac{4}{3\pi}\sin\frac{3x}{2} + \frac{4}{5\pi}\sin\frac{5x}{2}.$$

Substituting Eq. (4.1) in integral equation (4.2) and equating coefficients of identical powers of θ_0, we obtain an equation for determination of $\varphi_n(x)$. Thus, the equation for $\varphi_1(x)$ has the form

$$\varphi_1(x) - \frac{1}{\pi}\int_0^{\pi}K(x, y)\varphi_1(y)\,dy = \psi_1(x).$$

This equation is solved just like the one for $\varphi_0(x)$:

$$\varphi_1(x) = -(2/\pi)b_1\sin(x/2).$$

In the equation for $\varphi_2(x)$ a contribution is produced by

$$\frac{1}{\pi}\int_0^{\theta_0}K(x, y)\varphi_0(y)\,dy.$$

Using the expansion of $K(x, y)$ at $|y| < \theta_0$ and the form of $\varphi_0(x)$, we obtain

$$\varphi_2(x) = \left(\frac{a_2}{2\pi} - \frac{2}{\pi}b_2 - \frac{1}{16\pi} - \frac{1}{4\pi}b_0\right)\sin\frac{x}{2} + \frac{8a_2}{3\pi}\sin\frac{3x}{2} - \frac{4a_2}{5\pi}\sin\frac{5x}{2}.$$

From the condition $C_0(\tau)=0$ we define the first terms in the expansion in θ_0 of the coefficient $A_0(\tau)$:

$$A_0(\tau) = (\tau - 1)/4 + O(\theta_0^3).$$

From this it follows that

$$\int_0^{\theta_0} \varphi_0(y)\,dy = O(\theta_0^4), \quad \int_0^{\theta_0} y\varphi_0(y)\,dy = O(\theta_0^5), \quad \varphi_1(x) \equiv 0.$$

Using these relationships, we write the equation for determination of $\varphi_3(x)$

$$\varphi_3(x) - \frac{1}{\pi}\int_0^{\pi} K(x, y)\,\varphi_3(y)\,dy = \psi_3(x).$$

The solution of this equation has the form

$$\varphi_3(x) = \left(\frac{a_3}{2\pi} - \frac{2}{\pi}b_3\right)\sin\frac{x}{2} + \frac{8a_3}{3\pi}\sin\frac{3x}{2} - \frac{4a_3}{5\pi}\sin\frac{5x}{2}.$$

The desired function $\varphi(x, \tau)$ can be written in the form

$$\varphi(x, \tau) = \frac{4}{3\pi}(2\tau - 1)\sin\frac{3x}{2} + \frac{4}{5\pi}(1 - \tau)\sin\frac{5x}{2} + O(\theta_0^4).$$

Now knowing $\varphi_3(x)$, we can refine the value of $A_0(\tau)$:

$$A_0(\tau) = \frac{\tau - 1}{4} + O(\theta_0^4).$$

From Eq. (3.1), we define $\theta_0(\tau)$:

$$\theta_0(\tau) = \sqrt{\frac{3}{2}\tau} + O(\tau^{3/2}).$$

5. Using Eqs. (3.2), (3.3) we define the pressure on the contact spot:

$$p = \frac{3}{\sqrt{2\pi}}\frac{\gamma v^2}{\sqrt{\cos\theta - \cos\theta_0}} + O(1). \tag{5.1}$$

With Eq. (3.4) we than calculate the collision force

$$F = 6\sqrt{3}\gamma a^{3/2} v^{5/2} t^{1/2} + O(t). \tag{5.2}$$

Using Eqs. (2.1), (5.1) we find that the major term in the asymptote of Φ is not positive on the contact spot as $t \to 0$. Physically, this means that particles lying on the contact spot flow away from the point $\theta = 0$, $\rho = a$.

We now apply a Kelvin transform to the function $\Phi(\rho, \theta, t)$ which is harmonic within the sphere. Then the function

$$w(\rho, \theta, t) = \begin{cases} \Phi(\rho, \theta, t), & \rho \leqslant a, \\ -\frac{a}{\rho}\Phi\left(\frac{a^2}{\rho}, \theta, t\right), & \rho \geqslant a \end{cases}$$

will be harmonic over all space with the surface Σ being a special case. On one side of Σ we have the Neumann condition, and on the other, a condition of the third sort:

$$w_\rho + w/a = (a - vt)/\cos\theta - a, \quad \rho = a + 0, \quad 0 \leqslant \theta < \theta_0 : \Sigma_+,$$
$$w_\rho = (a - vt)/\cos\theta - a, \quad \rho = a - 0, \quad 0 \leqslant \theta < \theta_0 : \Sigma_-.$$

We turn to a new function $q = \partial w/\partial\rho$, with q being harmonic over all space and equal to zero at infinity. By the intensified principle of the maximum, q cannot achieve its maximum value within the region of its definition. Therefore

$$q \leqslant \max\left(\max_{\Sigma_-} q, \max_{\Sigma_+} q\right), \quad \rho = a, \quad \theta_0 < \theta \leqslant \pi.$$

On Σ_- max $q = vt/2$, while on Σ_+ max $q = vt/2 + \max(\Phi/a)$. Since $\Phi|_\Sigma \leq 0$, then $q \leq vt/2$ at $\rho = a$, $\theta_0 < \theta \leq \pi$. This inequality may be intensified to

$$q|_{\rho=a} \leqslant (a - vt)/\cos\theta - a, \quad \theta_0 < \theta \leqslant \pi/2.$$

Thus, the solution constructed satisfies condition (2.3).

Note 1. It is known that the pressure p is a subharmonic function within Ω_0 [5]. Since $p \geq 0$ on Σ and $p = 0$ on Γ by definition, consequently $p \geq 0$ within Ω_0, i.e., there are no rarefaction zones in the flow.

Note 2. Due to the axial symmetry of the problem, the center of the mass can move only along the z axis upon collision, and its motion can be determined from Newton's law:

$$m d^2 Z_d / dt^2 = F(t),$$

where m is the droplet mass, Z_d is the displacement of the droplet's center of mass along the z axis, and F(t) is the force specified by Eq. (5.2). It can be seen that $Z_d = O(t^{5/2})$ as $t \to 0$, i.e., the motion of the center of mass is negligibly small in the initial stage of collision.

Note 3. By applying an additional condition ($\Phi \leq 0$ on Σ), the problem of Eqs. (2.2), (2.3) can be formulated as one with single-sided inequalities:

$$\Delta w^* = 0 \text{ in } \Omega_0, \quad \begin{cases} w_\rho^* - g^* \geqslant 0 \\ w^* \geqslant 0 \\ w^* (w_\rho^* - g^*) = 0 \end{cases} \quad \text{on } \partial\Omega_0,$$

where $w^* = -\Phi$, $g^* \in H^{1/2}(\Omega_0)$. We will seek a solution of this problem within $H^1(\Omega_0)$. We define the set $U_\partial = \{v | v \in H^1(\Omega_0),\ v \geqslant 0 \quad \text{on } \partial\Omega_0\}$, the continuous liner function

$$L(v) = \int_{\partial\Omega_0} g^* v d\Gamma, \quad v \in H^1(\Omega_0)$$

and the bilinear, symmetric, continuous form, coercive on $H^1(\Omega_0)$

$$\pi(u, v) = \int_{\Omega_0} \nabla u \nabla v d\Omega, \quad v, u \in H^1(\Omega_0).$$

With the aid of Green's formula it can be shown that our problem with single-sided inequalities is equivalent to the variation inequality

$$\pi(w^*, v - w^*) \geqslant L(v - w^*), \ \forall v \in U_\partial.$$

It is known [6] that a solution of this inequality exists and is unique in $H^1(\Omega_0)$.

The problem of entry of a blunt body into a liquid can also be formulated as a variation inequality, but the question of existence and uniqueness of its solution is not then trivial, since the form $\pi(u, v)$ for unlimited regions is not coercive.

6. The flow asymptote constructed above loses meaning in some immediate vicinity of the line $\theta = \theta_0(t)$, $\rho = a$, the size of which is of the order of t as $t \to 0$. In this vicinity the pressure and velocity are infinite, and a significant role is played by the compressibility, viscosity, and surface tension of the liquid. Thus, within this region it is necessary to construct an "internal expansion," which describes the flow fine structure only near the contact line.

Of special interest in the problem of droplet collision are the pressure value on the contact spot, the point of application of maximum loading, and the duration of this action. The basic studies of this question (see review [7]) employ the model of an ideal compressible liquid, with neglect of surface tension and external mass forces. The model of an ideal incompressible liquid was used in [8] for numerical calculation of the central collision of two droplets in the absence of mass forces. As $t \to 0$ the accuracy of the numerical calculations decreases significantly, so that an analytical study of the initial collision stage which determines the entire "history" of the motion is of great importance. Compressibility must be considered only at the very first moment of time [9, 10], before the compression wave front has departed beyond the limits of the contact spot. This stage of droplet collision with a solid plane was studied in [11].

We will now calculate the Weber number $W = a\gamma v^2/\sigma$, Froude number $Fr = v^2/ag$, Reynolds number $Re = av/\nu$ and Mach number $M = v/c$ for this process, where a is the droplet radius, γ is the liquid density, v is the collision velocity, σ is the surface tension coefficient, g is the acceleration of gravity, ν is the kinematic viscosity coefficient, and c is the speed of sound in the liquid. At $a = 3 \cdot 10^{-3}$ m, $v = 100$ m/sec, $\nu = 10^{-4}$ m^2/sec, $c = 1500$ m/sec, $\sigma = 72.58 \cdot 10^{-3}$ J/m^2, $\lambda = 10^3$ kg/m^3, $g = 9.81$ m/sec^2 we obtain W, $Fr \sim 10^6$, $Re \sim 10^5$, $M \sim 10^{-2}$.

Thus, in this velocity range it can be expected that the forces of viscosity, surface tension, and gravity have an insignificant effect on the collision process, while the compressibility of the liquid can be neglected after the compression wave passes onto the droplet free surface.

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LITERATURE CITED

1. H. Wagner, "Über Stoss- und Gleitvorgänge an der Oberfläche von Flüssigkeiten," Z. Angew. Math. Mech., 12, No. 4 (1932).
2. V. V. Pukhnachev, "Linear approximation in the problem of blunt body entry into water," in: Dynamics of a Continuous Medium [in Russian], No. 38, Inst. Gidrodin. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1979).
3. V. I. Nalimov and V. V. Pikhnachev, Nonstationary Motions of an Ideal Liquid with a Free Surface [in Russian], Novosibirsk (1975).
4. Ya. S. Uflyand, The Paired Equation Method in Mathematical Physics Problems [in Russian], Leningrad (1977).
5. N. V. Kochin, I. A. Kibel, and N. V. Roze, Theoretical Hydromechanics [in Russian], Part I, Moscow (1963).
6. J. L. Lyons, Optimum Control of Systems Described by Partial Derivatives [Russian translation], Moscow (1972).
7. A. L. Gonor and V. Ya. Yakovlev, "Some results concerning the theory of droplet collision on a solid surface," in: Jet and Detached Flows [in Russian], Izd. Inst. Mekh. Mosk. Gos. Univ., Moscow (1979).
8. V. E. Petrenko, "Methods for calculation of flows of viscous incompressible liquids with high deformations," in: Oscillations of Liquid-Filled Elastic Constructions [in Russian], Novosibirsk (1973).
9. V. A. Eroshin, N. I. Romanenkov, et al., "Hydrodynamic forces upon collision of blunt bodies with a compressible liquid surface," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza (1980).
10. A. G. Gorshkov and É. I. Grigolyuk, "The problem of collision of elastic shells of rotation with an incompressible liquid," in: Oscillations of Liquid-Filled Elastic Constructions [in Russian], Novosibirsk (1973).
11. G. J. Heymann, "High-speed impact between a liquid drop and a solid surface," J. Appl. Phys., 40, No. 13 (1969).

DEVELOPMENT OF INITIAL PERTURBATIONS OF THE EXTERNAL BOUNDARY OF AN EXPANDING GAS – LIQUID RING

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In studies of surface phenomena related to underwater explosions, in particular, in studying the process of splash dome formation, the development of perturbations in the initial stage of free surface motion is of interest. A convenient model to use in such studies is that of the flow occurring upon explosion of a cylindrical charge in a cylindrical liquid ring, where the free surface form coincides with that of the charge. The stability of an expanding liquid ring has been considered in a number of studies.

Thus, assuming an ideal incompressible liquid, [1] considered the stability of initial perturbations of a thin liquid ring expanding inertially. It was shown that in the general case such motion is unstable; introduction of surface tension has a stabilizing effect on harmonics. But in the case where the liquid motion takes place under the stimulus of impulse loading, commencement of liquid motion is preceded by exit of a shock wave onto the liquid surface, as a result of which the reflected unloading wave destroys the continuity of the liquid medium. Thus in this case the validity of using stability estimates obtained in problems concerning expansion of a continuous liquid ring is questionable.

The present study is an experimental investigation of the development of initial perturbations on the external surface of an expanding gas–liquid ring. Such a flow was realized in the following manner. Along the

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